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Symmetry of steady periodic surface water waves with vorticity

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For large classes of vorticities we prove that a steady periodic gravity water wave with a monotonic profile between crests and troughs must be symmetric. The analysis uses sharp maximum principles for elliptic partial differential equations.

1. Introduction

We discuss two-dimensional periodic gravity waves which propagate steadily on a shearing water flow over an impermeable and flat bed. These are plane waves on the water surface, with no variation along their crests, and for which the momentum and gravity forces are dominant – we neglect capillarity and viscosity. The inviscid setting is realistic since the time scales/length scales associated with an adjustment of the flow conditions due to viscosity are long compared with the wave period/wavelength (Johnson 1997). Also, for water waves with wavelengths much above 1.74 cm the effect of surface tension is very small and can be neglected (Lighthill 1978). While most studies of water waves are restricted to irrotational flows, there are many circumstances where vorticity plays an essential role. Waves with vorticity are commonly seen in nature. For example, if the water is shallow and waves are long, the shear caused by currents can become a dominant feature of the wave motion (Peregrine 1976). Since the waves are long compared with the water depth, in this case it is the existence of a non-zero mean vorticity that is important rather than its specific distribution (Teles da Silva & Peregrine 1988). On areas of the continental shelf and in many coastal inlets the most significant currents are the tides (Jonsson 1990). They are the most regular currents and the assumption of constant vorticity is realistic (Swan, Cummins & James 2001). Constant vorticity, however, does not give a good description of wind drift currents, like the major ocean currents such as the Gulf Stream (Jonsson 1990). Also, out-flowing waves at the mouth of an estuary generally exhibit a non-uniform vorticity distribution (Swan *et al.* 2001). Even in appropriate experimental situations where it is to regard the flow as being irrotational, it is important to know that the main features of the irrotational case also occur for flows with small vorticity. For example, the study of waves propagating into still water is usually performed within the irrotational framework. However, a uniform vorticity always tends to be generated at the free surface of a progressive wave advancing into still water and this vorticity is propagated slowly downwards from the free surface, as has been verified experimentally (Longuet-Higgins 1953, 1960).

The existence of regular wave trains for irrotational flows is well established (Amick & Toland 1981; Keady & Norbury 1978). In 1809 Gerstner constructed an explicit example of a periodic travelling wave in water of infinite depth with a particular non-zero vorticity – see (Constantin 2001) for a discussion. It is intriguing to speculate that there might be also exact solutions for finite depth. However, it seems that no exact solution for gravity waves on water of finite depth is known, and one of the most significant advances has been the development of computationally efficient numerical models by Teles da Silva & Peregrine (1988). Numerical investigations for general vorticity (Baddour & Song 1998) indicate that regular wave trains are possible only in the case of constant vorticity. However, the existence of regular wave trains for flows with non-constant vorticity can be established (Constantin & Strauss 2002, 2004). The result of the present study confirms that regular wave trains for flows with vorticity are quite ubiquitous.†

THEOREM 1. *Consider a steady periodic wave train propagating over the flat bed* $y=0$ *with relative mass flux* $-m<0$. *Assume that the wave profile* $y=\eta(x)$ *is monotonic between crests and troughs. Then the wave is symmetric, provided*

$$\gamma'(s)\max_{x\in\mathbb{R}}\eta^2(x) < \pi^2, \qquad 0\leqslant s\leqslant m, \tag{1}$$

where $\gamma\in C^1([0,m],\mathbb{R})$ *is the vorticity function of the flow.*

There are two important situations in which (1) holds. If the vorticity is decreasing with depth (that is, $\gamma'\leqslant 0$), then our result excludes non-symmetric steady periodic gravity waves with profiles that are monotonic between crests and troughs. For other vorticity distributions the above conclusion holds provided the maximal elevation of the water above the flat bottom is small enough. In the case of constant vorticity our investigation may be regarded as a small step towards finding theoretical confirmation of the very striking numerical results given in Teles da Silva & Peregrine (1988). Note that in the special case $\gamma\equiv 0$ (irrotational flow) Garabedian (1965) proved the symmetry of steady periodic gravity waves with one local maximum and one local minimum per wavelength on every streamline except for the flat bottom – we refer to (Toland 2000) for a simpler proof. In contrast to Garabedian's and Toland's variational approaches, our method is based on symmetrization and sharp maximum principles for subsolutions to second-order elliptic partial differential equations (Serrin 1971; Gidas, Ni & Nirenberg 1979). Considerations similar to ours have been presented in Okamoto & Shoji (2001) for the simpler case of irrotational flows.

In § 2 we formulate the physical problem in mathematical terms. Section 3 contains the proof of the Theorem.

2. Preliminaries

In this section we recall the governing equations for the propagation of two-dimensional gravity waves on water and we give a reformulation suitable for our purposes.

The motion is identical in any direction orthogonal to the direction of propagation of the wave. Therefore it suffices to analyse a cross-section of the flow, perpendicular

† This does not exclude the possibility of the existence of non-symmetric waves as it appears (Smith 1976) that large asymmetric waves are possible on currents in deep water. Note, however, that the asymmetric waves calculated by Smith (1976) were not periodic.

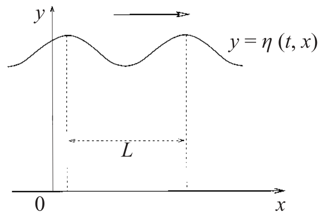

FIGURE 1. The problem under consideration.

to the crest line. We choose Cartesian coordinates (x, y) so that the horizontal x-axis is in the direction of wave propagation, the y-axis points vertically upwards and the origin lies on the flat bed (see figure 1). Let $y=\eta(t,x)$ denote the free surface of the water and let $(u(t,x,y), v(t,x,y))$ be the velocity field of the flow. In the study of sea waves it is appropriate to regard water as an incompressible fluid (Lighthill 1978). Thus, conservation of mass implies that

$$u_x + v_y = 0. \tag{2}$$

Furthermore, since gravity waves should be considered inviscid (Teles da Silva & Peregrine 1988), the flow is governed by Euler's equation

$$\left.\begin{aligned} u_t + uu_x + vu_y &= -P_x, \\ v_t + uv_x + vv_y &= -P_y - g, \end{aligned}\right\} \tag{3}$$

where $P(t,x,y)$ denotes the fluid pressure and g is the gravitational constant of acceleration. Neglecting the effects of surface tension, the dynamic boundary condition

$$P = P_0 \quad \text{on} \quad y = \eta(t,x), \tag{4}$$

P_0 being the constant atmospheric pressure, decouples the motion of the air from that of the water (Crapper 1984). The kinematic boundary conditions

$$v = \eta_t + u\eta_x \quad \text{on} \quad y = \eta(t,x), \tag{5}$$

and

$$v = 0 \quad \text{on} \quad y = 0, \tag{6}$$

express, respectively, that the same particles always form the free surface, and that it is impossible for the water to penetrate the rigid bed (Johnson 1997). Summarizing, (2)–(6) form the governing equations for two-dimensional gravity water waves.

Given $c>0$, we consider wave trains travelling at speed c. That is, we assume that the (x,t)-space–time dependence of the free surface, of the pressure, and of the velocity field has the form $(x-ct)$, and that P, u, v, and η all display a periodic dependence upon the x-variable of minimal period, say, $L>0$. The change of frame $(x-ct, y)\mapsto(x,y)$ eliminates time from the problem. In the new moving reference frame the wave is stationary and the flow is steady. Concerning regularity, we impose that $\eta\in C^3(\mathbb{R})$ and $(P,u,v)\in C^1(\overline{D}_\eta)\times C^2(\overline{D}_\eta)\times C^2(\overline{D}_\eta)$, where $\overline{D}_\eta := \{(x,y)\in\mathbb{R}^2; 0\leqslant y\leqslant \eta(x)\}$ is the closure of the fluid domain. Experimental evidence indicates that for wave patterns that are not near the spilling or breaking state, the propagation speed of the surface wave is in general considerably larger than the speed of each individual water particle (Banner & Peregrine 1993; Lighthill 1978). In view of this, we require that $u<c$ throughout the whole fluid body.

The mass flux across $x = x_0$ relative to the uniform flow at speed c is

$$\int_0^{\eta(x_0)} [u(x_0, y) - c]\,\mathrm{d}y.$$

Using (5) and (6), this expression can be seen to be independent of x_0. This leads us to define the relative mass flux as

$$M := \int_0^{\eta(x)} [u(x, y) - c]\,\mathrm{d}y, \qquad x \in \mathbb{R}.$$

Note that $M < 0$, since $u, < c$ throughout the fluid. To facilitate notation, we set $m := -M > 0$.

It is convenient to formulate the water wave problem in terms of the (relative) stream function $\psi(x, y)$ defined by

$$\psi_x = -v, \quad \psi_y = u - c. \tag{7}$$

The stream function is given explicitly by

$$\psi(x, y) = \psi_0 + \int_0^{y} [u(x, \xi) - c]\,\mathrm{d}\xi,$$

and is uniquely determined up to the constant $\psi_0 \in \mathbb{R}$. The boundary conditions (5) and (6) show that ψ is constant on the free surface $y = \eta(x)$ and on the flat bed $y = 0$, respectively. If we normalize ψ to be zero on the free surface (this amounts to choosing $\psi_0 = m$), the definition of the relative mass flux forces $\psi = m$ on $y = 0$. Therefore, ψ is a strictly decreasing function of the height y. Since ψ is of class C^2, the assumption $u - c = \psi_y < 0$ ensures by the implicit function theorem that the level sets of ψ are locally C^2-curves. However, relation (7) shows that $\psi_y < 0$ inside the fluid domain so that ψ is a strictly decreasing function of the height y. Combining all this, we conclude that the streamlines $\psi = \text{constant} \in [0, m]$ give a foliation of the fluid domain, the wave profile $y = \eta(x)$ being the streamline $\psi = 0$, and the flat bed $y = 0$ corresponding to $\psi = m$, respectively. Let $\omega := v_x - u_y$ denote the vorticity of the flow. Then $\Delta\psi = -\omega$. From the above definition of ψ we obtain

$$(u - c)\psi_x + v\psi_y = 0. \tag{8}$$

On the other hand, taking the curl of the Euler equation (3) yields

$$(u - c)\omega_x + v\omega_y = 0. \tag{9}$$

Relation (8) shows that $(u - c, v)$ at (x_0, y_0) points in the direction of the tangent $\tau(x_0, y_0)$ to the level curve $\mathscr{C}$ of ψ passing through (x_0, y_0). Therefore, (9) forces that $\nabla\omega(x_0, y_0)$ is either zero or orthogonal to $\tau(x_0, y_0)$. In both cases this implies that ω is constant on $\mathscr{C}$, as shown by a simple differentiation along this curve. Since ω is constant along the level curves of ψ, we have that, at least locally, ω is a function of ψ. Since $u < c$, one can prove that there is $\gamma \in C^1([0, m], \mathbb{R})$ such that $\omega = \gamma(\psi)$ throughout the fluid (Constantin & Strauss 2004). The vorticity function γ is a measure of the strength of the vorticity.

It follows from (3) and (7) that the total energy

$$E := \frac{(u - c)^2 + v^2}{2} + gy + P - \Gamma(-\psi)$$

is constant within fluid, where

$$\Gamma(s) := \int_0^s \gamma(-\xi)\, \mathrm{d}\xi \ \text{ for } s \in [0, m].$$

This is Bernoulli's Law. In the expression for E, the sum of the first three terms represents the total mechanical energy of the flow, which can be measured at any location with a piezometer. Here $\frac{1}{2}[(c-u)^2 + v^2]$ is the kinetic energy (energy of motion), gy is the gravitational potential energy (energy of position) and P is the energy of fluid pressures (exerted on a particle by the surrounding fluid acting upon it). Bernoulli's Law shows, in view of (7), that the kinematic boundary condition (4) is equivalent to

$$|\nabla\psi|^2 + 2gy = C \quad \text{on} \quad y = \eta(x),$$

where $C := 2(E - P_0)$. Summarizing, we obtain the following nonlinear elliptic free boundary value problem

$$\left.\begin{array}{lll} \Delta\psi = -\gamma(\psi) & \text{in} & 0 < y < \eta(x), \\ |\nabla\psi|^2 + 2gy = C & \text{on} & y = \eta(x), \\ \psi = 0 & \text{on} & y = \eta(x), \\ \psi = m & \text{on} & y = 0, \end{array}\right\} \tag{10}$$

to be satisfied for $\eta \in C^3(\mathbb{R})$ and $\psi \in C^2(\overline{D}_\eta)$, both L-periodic in the x-variable. The fact that the problem (10) is equivalent to the governing equations (2)–(6) is proved in Constantin & Strauss (2004).

3. Proof of the theorem

Our approach is based on a device of moving parallel lines to a critical position and then showing that the solution is symmetric about the limiting line the moving plane method – see (Serrin 1971; Gidas *et al.* 1978). Throughout the analysis we employ sharp maximum principles for elliptic partial differential equations, which we present now as a lemma in a form suitable for our purposes.

LEMMA 1. *Let Ω be the open domain in the (x, y)-plane lying between the graph $y = f(x)$ of a positive continuous function $f : [a, b] \to (0, \infty)$ and the horizontal line $y = 0$. That is, $\Omega = \{(x, y) \in \mathbb{R}^2 : \ a < x < b, 0 < y < f(x)\}$. For functions $b_1, b_2, c \in C(\overline{\Omega}, \mathbb{R})$ such that $c(x, y) \leqslant 0$ throughout $\overline{\Omega}$, define the elliptic operator*

$$\mathscr{L} = \partial_x^2 + \partial_y^2 + b_1(x, y)\, \partial_x + b_2(x, y)\, \partial_y + c(x, y).$$

(*i*) *If $w \in C^2(\Omega) \cap C(\overline{\Omega})$ is such that $\mathscr{L}w \leqslant 0$ in Ω and $w \geqslant 0$ on the boundary $\partial\Omega$ of Ω, then $w > 0$ in Ω unless $w \equiv 0$ in $\overline{\Omega}$.*

(*ii*) *Let $w \in C^2(\Omega) \cap C(\overline{\Omega})$. Suppose that $w \geqslant 0$ in $\overline{\Omega}$, $\mathscr{L}w \leqslant 0$ in Ω, and $w = 0$ at some point $Q \in \partial\Omega$. If Ω satisfies an interior sphere condition (That is, there exists a small open ball contained in Ω with Q on its boundary) at Q, then the outer normal derivative $\partial w/\partial \nu$ of w at Q, if it exists, satisfies the strict inequality $\partial w/\partial \nu < 0$, unless $w \equiv 0$ on $\overline{\Omega}$.*

(*iii*) *Assume that f is twice continuously differentiable and let T be the line containing the normal to $y = f(x)$ at some point $Q \in \partial\Omega$. Let Ω_0 then denote the portion of Ω lying on some particular side of T. Suppose that $w \in C^2(\overline{\Omega_0})$ satisfies $\mathscr{L}w \leqslant 0$ in Ω_0, while also $w \geqslant 0$ in Ω_0 and $w = 0$ at Q. Then either $\partial w/\partial \mu > 0$ or $\partial^2 w/\partial \mu^2 > 0$ at Q unless $w \equiv 0$ on $\overline{\Omega_0}$, where μ is any direction at Q which enters Ω non-tangentially.*

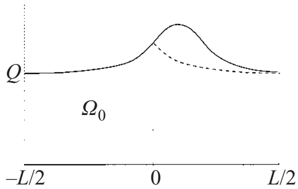

FIGURE 2. Case (*a*).

Assertions (i) and (ii) are respectively the Weak and the Hopf Maximum Principle, whereas (iii) is a version of the Edge Point Lemma proved in Fraenkel (2000).

Proof of the theorem. For simplicity we choose the trough of the surface wave $y=\eta(x)$ at $x=\pm L/2$. Let $\eta_0=\max_{x\in\mathbb{R}}\{\eta(x)\}>0$. In view of (1), there is some $\delta\in(0,1/2)$ such that

$$\eta_0^2\,\gamma'(s)\leqslant \pi^2(1-2\delta)^2, \qquad 0\leqslant s\leqslant m. \tag{11}$$

Let us now introduce the function

$$\alpha(y)=\sin\,\pi\left[(1-2\delta)\frac{y}{\eta_0}+\delta\right], \qquad 0\leqslant y\leqslant \eta_0.$$

The argument of the sine in the definition of $\alpha(y)$ belongs to $[\pi\delta,\pi-\pi\delta]$ so that $0<\sin(\pi\delta)\leqslant\alpha(y)\leqslant 1$ for all $y\in[0,\eta_0]$.

For $x_*\in(-L/2,0]$ we define

$$D_*=\{(x,y)\in\mathbb{R}^2\,:\,-d<y<\eta(x)\text{ for }-L/2<x<x_*\}.$$

The map $(x,y)\mapsto(2x_*-x,y)$ reflects the domain D_* in the line $x=x_*$ into a domain D_*^R. Since $x=-L/2$ is the location of the wave trough, the monotonicity property of the free surface ensures the existence of some $\varepsilon>0$ small enough such that the function $x\mapsto\eta(x)$ is non-decreasing on $(-L/2,-L/2+\varepsilon)$. Therefore D_*^R is a subset of the fluid domain

$$D=\{(x,y)\in\mathbb{R}^2\colon\, 0<y<\eta(x)\}$$

for all $x_*\in(-L/2,-L/2+\varepsilon)$. As we increase x_* from $-L/2$ there is some maximal $x_0\in(-L/2,0]$ such that D_*^R is included in D for all $x_*\in(0,x_0)$. Note that D_0^R, corresponding to $x_*=x_0$, is still a subset of D. At $x=x_0$ one of the following three situations occurs:

(*a*) $x_0=0$;

(*b*) the vertical line $x=x_0$ is normal to the free surface $y=\eta(x)$ at the crest point $(x_0,\eta(x_0))$;

(*c*) D_0^R is internally tangent to the boundary $y=\eta(x)$ at some point.

Let us first assume that (*a*) occurs – a typical case is depicted in figure 2.

Let $Q=(-L/2,\eta(-L/2))$ and define

$$w(x,y)=\frac{\psi(-x,y)-\psi(x,y)}{\alpha(y)}, \qquad -L/2\leqslant x\leqslant 0,\ 0\leqslant y\leqslant\eta(x),$$

where ψ is the stream function introduced in § 2. To state the theorem if (*a*) occurs, we claim that it suffices to show that $w\equiv 0$ in

$$\Omega_0=\{(x,y)\in\mathbb{R}^2\colon\, -L/2<x<0, 0<y<\eta(x)\}.$$

Then $\psi(-x,\eta(x))=\psi(x,\eta(x))$ for all $x\in[-L/2,0]$. Since the free surface $y=\eta(x)$ is given implicitely by $\psi=0$, we infer that $\psi(-x,\eta(x))=\psi(-x,\eta(-x))=0$ for all

$x \in [-L/2, 0]$. The injectivity of the function $y \mapsto \psi(x, y)$ for every fixed x, ensured by $\psi_y = u - c < 0$, yields $\eta(x) = \eta(-x)$ for every $x \in [-L/2, 0]$. Therefore the wave is symmetric.

To prove that $w \equiv 0$ in $\overline{\Omega_0}$, we proceed as follows. Observe that $w \in C^2(\overline{\Omega_0})$. The periodicity property of ψ implies $w = 0$ on $x = \pm L/2$. Moreover, $w = 0$ on the bottom $y = 0$, as $\psi = m$ there. Since $x_0 = 0$, we deduce that $(-x, \eta(x)) \in \overline{D}$ for all $x \in (-L/2, 0)$. Therefore $\psi(-x, \eta(x)) \geqslant 0$ for all $x \in (-L/2, 0)$, as $\psi \geqslant 0$ within the fluid. On the other hand, $\psi(x, \eta(x)) = 0$ for $x \in (-L/2, 0)$ in view of (10). Hence $w(x, \eta(x)) \geqslant 0$ for all $x \in (-L/2, 0)$. Thus $w \geqslant 0$ on the boundary $\partial\Omega_0$ of Ω_0. At this point, note that a simple calculation confirms the identity

$$\Delta\left(\frac{w_0}{\alpha}\right) + 2\frac{\alpha_y}{\alpha}\partial_y\left(\frac{w_0}{\alpha}\right) = \frac{\Delta w_0}{\alpha} - \frac{\alpha_{yy}}{\alpha}\frac{w_0}{\alpha}$$

for all C^2-functions $w_0(x, y)$ and $\alpha(y)$. We choose

$$w_0(x, y) = \psi(-x, y) - \psi(x, y), \qquad -L/2 \leqslant x \leqslant 0, 0 \leqslant y \leqslant \eta(x),$$

whereas $\alpha(y)$ is given in the beginning of the proof. Since $\Delta\psi = -\gamma(\psi)$ throughout the fluid and $w_0 = \alpha w$, the above identity becomes

$$\Delta w + 2\frac{\alpha_y}{\alpha}\partial_y w + \frac{\tilde{\gamma}}{\alpha} - \frac{\pi^2(1-2\delta)^2}{\eta_0^2}w = 0, \qquad -L/2 \leqslant x \leqslant 0, 0 \leqslant y \leqslant \eta(x),$$

where $\tilde{\gamma}(x, y) = \gamma(\psi(-x, y)) - \gamma(\psi(x, y))$. The mean value theorem ensures the existence of some $s_0(x, y) \in (0, m)$ such that $\tilde{\gamma}(x, y) = \gamma'(s_0)[\psi(-x, y) - \psi(x, y)]$. It follows that

$$\Delta w + 2\frac{\alpha_y}{\alpha}\partial_y w + w\left(\gamma'(s_0) - \frac{\pi^2(1-2\delta)^2}{\eta_0^2}\right) = 0, \qquad -L/2 \leqslant x \leqslant 0,\ \ 0 \leqslant y \leqslant \eta(x).$$

In view of the above equation and (11), the fact that $w \geqslant 0$ on $\partial\Omega_0$ ensures by the Lemma, part (i), that either $w > 0$ in Ω_0 or $w \equiv 0$ on $\overline{\Omega_0}$. Noticing that $w = 0$ at Q, part (iii) of the Lemma (with $T = \{x = -L/2\}$) yields $w \equiv 0$ in Ω_0 if at the point Q all partial derivatives of w of order less than or equal to 2 are equal to zero. We now show that this is the case. First, the way we defined the periodic function w guarantees that $w_y(Q) = w_{xx}(Q) = w_{yy}(Q) = 0$ since $w(Q) = 0$. Differentiating the relation $\psi(x, \eta(x)) = 0$, we obtain $\psi_x + \psi_y\eta' = 0$ on $y = \eta(x)$. But $\eta'(-L/2) = 0$ since Q is the wave trough, so that $\psi_x(Q) = 0$ and $w_x(Q) = -2\,\psi_x(Q)/\alpha(Q) = 0$. It remains to show that $w_{xy}(Q) = 0$. Differentiating the nonlinear boundary condition on $y = \eta(x)$ from (10) with respect to x, we obtain

$$\psi_x(\psi_{xx} + \psi_{xy}\eta') + \psi_y(\psi_{xy} + \psi_{yy}\eta') + g\eta' = 0 \quad \text{on} \quad y = \eta(x).$$

Evaluating this at the wave trough Q, where $\eta' = \psi_x = 0$, we obtain $\psi_y(Q)\psi_{xy}(Q) = 0$. Since by assumption $\psi_y = u - c < 0$, we must have $\psi_{xy}(Q) = 0$. But

$$w_{xy}(Q) = -2\frac{\psi_{xy}(Q)}{\alpha(Q)} - \frac{\alpha_y(Q)}{\alpha(Q)}w_x(Q),$$

and we conclude that $w_{xy}(Q) = 0$ since we already know that $w_x(Q) = 0$. Therefore the wave is symmetric if case (a) occurs.

Let us now analyse the alternative (b) – see figure 3. Note that we must have $x_0 \leqslant 0$ and, since $x_0 = 0$ is precisely case (a), we may assume that $x_0 < 0$. The defining property of x_0 ensures that the domain D_0^R, obtained by reflecting $D_0 = \{(x, y) \in \mathbb{R}^2$:

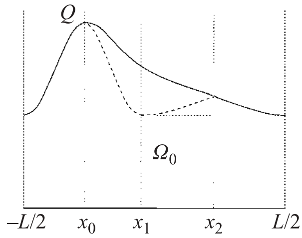

FIGURE 3. Case (b).

$-L/2 < x < x_0,\ 0 < y < \eta(x)\}$ in the line $x = x_0$ by means of the transformation $(x, y) \mapsto (2x_0 - x, y)$, is contained within the fluid domain D. Since $(x_0, \eta(x_0))$ is the wave crest, the wave profile $y = \eta(x)$ is decreasing on $[x_0, L/2]$. Therefore, letting $x_1 = 2x_0 + L/2$ and $x_2 = x_0 + L/2$, the reflection via the transformation $(x, y) \mapsto (2x_2 - x, y)$ of the domain $\{(x, y) \in \mathbb{R}^2 : x_2 < x < L/2,\ 0 < y < \eta(x)\}$ in the line $x = x_2$, is also contained within D. Observe that this reflection maps the line $\{x = L/2\}$ into $\{x = x_1\}$. We now define

$$w(x, y) = \begin{cases} \dfrac{\psi(x, y) - \psi(2x_0 - x, y)}{\alpha(y)}, & x_0 \leqslant x \leqslant x_1,\ 0 \leqslant y \leqslant \eta(2x_0 - x), \\[2ex] \dfrac{\psi(x, y) - \psi(2x_2 - x, y)}{\alpha(y)}, & x_1 \leqslant x \leqslant x_2,\ 0 \leqslant y \leqslant \eta(2x_2 - x), \end{cases}$$

and we claim that it suffices to show that $w \equiv 0$ on the closure of the domain

$$\Omega_0 = \{(x, y) \in \mathbb{R}^2 : x_0 < x < x_2,\ 0 < y < \tilde{\eta}(x)\}.$$

Here

$$\tilde{\eta}(x) = \begin{cases} \eta(2x_0 - x), & x_0 \leqslant x \leqslant x_1, \\ \eta(2x_2 - x), & x_1 \leqslant x \leqslant x_2. \end{cases}$$

Indeed, $w \equiv 0$ on $\overline{\Omega_0}$ implies that $\psi(2x_0 - x, \eta(2x_0 - x)) = \psi(x, \eta(2x_0 - x))$ for $x \in [x_0, x_1]$ and $\psi(2x_2 - x, \eta(2x_2 - x)) = \psi(x, \eta(2x_2 - x))$ for $x \in [x_1, x_2]$. Since $\psi_y = u - c < 0$ throughout $\overline{D}$ and the implicit equation of the free surface is $\psi(x, \eta(x)) = 0$, we deduce that $\eta(x) = \eta(2x_0 - x)$ for $x \in [-L/2, x_1]$ and $\eta(x) = \eta(2x_2 - x)$ for $x \in [x_1, L/2]$. That is, the wave profile $y = \eta(x)$ is symmetric with respect to $x = x_0$ on $[-L/2, x_1]$ and with respect to $x = x_2$ on $[x_1, L/2]$. But the profile is supposedly monotonic between crest and trough, that is, on each of the intervals $[-L/2, x_0]$ and $[x_0, L/2]$. This contradiction eliminates the possibility $x_0 < 0$. Therefore $x_0 = 0$ and the analysis we pursued in case (a) shows that the wave is symmetric.

To verify that $w \equiv 0$ in $\overline{\Omega_0}$ we will apply part (iii) of the Lemma with $Q = (x_0, \eta(x_0))$ and $T = \{x = x_0\}$. First, note that $w \in C^2(\overline{\Omega_0})$ and the function $\tilde{\eta}$ is twice continuously differentiable on $[x_0, x_2]$ with $\tilde{\eta}'(x_1) = 0$. Similar to case (a), we see that $w \geqslant 0$ on the top boundary of Ω_0, while $w = 0$ on the lateral and bottom boundaries of Ω_0. Also, just like in case (a), we see that

$$\Delta w + 2\,\frac{\alpha_y}{\alpha}\partial_y w + c(x, y)\,w = 0, \qquad (x, y) \in \Omega_0,$$

for some $c \in C(\overline{\Omega_0})$ with $c(x, y) \leqslant 0$ throughout $\overline{\Omega_0}$. Therefore, we may apply part (i) of the Lemma to infer that either $w > 0$ in Ω_0 or $w \equiv 0$ on $\overline{\Omega_0}$. Since $(x_0, \eta(x_0))$ is the crest of the wave, we have $\eta'(x_0) = 0$. An argument analogous to that pursued in the case of (a) confirms that at the point Q all partial derivatives of w of order less than

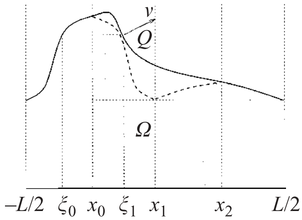

FIGURE 4. Case (c).

or equal to 2 are equal to zero. But $w=0$ at Q, so that by the Lemma, part (iii), we conclude that $w\equiv 0$ in $\overline{\Omega_0}$. Therefore symmetry also holds in case (b).

It remains to investigate the last alternative (c), corresponding to figure 4. Since $x_0=0$ characterizes case (a), we may assume that $x_0<0$ (clearly $x_0\leqslant 0$). Again, let $x_1=2x_0+L/2$ and $x_2=x_0+L/2$. Since the contact point $Q=(\xi_1,\eta(\xi_1))$ of the upper boundaries of D_0^R and D has to be located on the decreasing part of the wave profile, the reflection of the domain $\{(x,y)\in\mathbb{R}^2: x_2<x<L/2, 0<y<\eta(x)\}$ in the line $x=x_2$, achieved through the transformation $(x,y)\mapsto(2x_2-x,y)$, is contained in $\overline{D}$. This reflection maps the line $\{x=L/2\}$ into $\{x=x_1\}$.

Just like in the case of the alternative (b), to prove the symmetry of the wave it suffices to show that the function

$$w(x,y)=\begin{cases}\dfrac{\psi(x,y)-\psi(2x_0-x,y)}{\alpha(y)}, & x_0\leqslant x\leqslant x_1,\ 0\leqslant y\leqslant \eta(2x_0-x),\\[2mm] \dfrac{\psi(x,y)-\psi(2x_2-x,y)}{\alpha(y)}, & x_1\leqslant x\leqslant x_2,\ 0\leqslant y\leqslant \eta(2x_2-x),\end{cases}$$

is identically zero on the closure of the domain

$$\Omega=\{(x,y)\in\mathbb{R}^2: x_0<x<x_2, 0<y<\tilde{\eta}(x)\},$$

where, as before,

$$\tilde{\eta}(x)=\begin{cases}\eta(2x_0-x), & x_0\leqslant x\leqslant x_1,\\ \eta(2x_2-x), & x_1\leqslant x\leqslant x_2.\end{cases}$$

Observe that $w\in C^2(\Omega)$ and $\tilde{\eta}$ is twice continuously differentiable on $[x_0,x_2]$.

Let us prove that $w\equiv 0$ on $\overline{\Omega}$. Since $\psi\geqslant 0$ below the free surface $y=\eta(x)$ and $\psi=0$ on the free surface, we have that $w\geqslant 0$ on $y=\tilde{\eta}(x)$. The definition of w and the periodicity property of ψ ensure that $w=0$ on $\{x=x_0\}$ and on $\{x=x_2\}$. Also, $w=0$ on $y=0$ since $\psi=m$ on the flat bottom. Similarly to case (a), we have that

$$\Delta w+2\frac{\alpha_y}{\alpha}\partial_y w+c(x,y)w=0, \qquad (x,y)\in\Omega,$$

for some $c\in C(\overline{\Omega})$ with $c(x,y)\leqslant 0$ throughout $\overline{\Omega}$. Therefore, by part (i) of the Lemma, $w>0$ in Ω unless $w\equiv 0$ on $\overline{\Omega}$. We now claim that $\partial w/\partial\nu=0$ at Q, where ν is the outer normal to Ω at Q, implies $w\equiv 0$ on $\overline{\Omega}$. Indeed, the tangency property at Q ensures that Ω satisfies an interior sphere condition at Q. Moreover, note that $\eta(\xi_1)=\eta(2x_0-\xi_1)$ yields $\psi(\xi_1,\eta(2x_0-\xi_1))=\psi(\xi_1,\eta(\xi_1))=\psi(2x_0-\xi_1,\eta(2x_0-\xi_1))=0$ as $\psi=0$ on the free surface. Therefore $w=0$ at Q, and $(\partial w/\partial\nu)(Q)=0$ implies $w\equiv 0$ on $\overline{\Omega}$ in view of part (ii) of the Lemma. To check that $(\partial w\partial\nu)(Q)=0$, let $\xi_0=2x_0-\xi_1$. The tangency property at Q yields

$$\eta(\xi_0)=\eta(\xi_1) \quad \text{and} \quad \eta'(\xi_0)=-\eta'(\xi_1). \tag{12}$$

On the other hand, differentiating the relation $\psi(x, \eta(x)) = 0$ with respect to x, we obtain $\psi_x + \psi_y \eta' = 0$ on $y = \eta(x)$. Combining this with (12), we obtain that

$$\frac{\psi_x}{\psi_y}(\xi_0, \eta(\xi_0)) = -\frac{\psi_x}{\psi_y}(\xi_1, \eta(\xi_1)), \tag{13}$$

since $\psi_y = u - c < 0$ by assumption. Note also that (12) and the nonlinear boundary condition on $y = \eta(x)$ from (10) yield

$$|\nabla\psi|^2(\xi_0, \eta(\xi_0)) = |\nabla\psi|^2(\xi_1, \eta(\xi_1)). \tag{14}$$

Since $\psi_y = u - c < 0$ thoroughout $\overline{D}$, we deduce from (13) and (14) that

$$\psi_x(\xi_0, \eta(\xi_0)) = -\psi_x(\xi_1, \eta(\xi_1)) \quad \text{and} \quad \psi_y(\xi_0, \eta(\xi_0)) = \psi_y(\xi_1, \eta(\xi_1)).$$

This forces $\partial w/\partial \nu = 0$ at Q, if we take into account the definitions of ψ, α, ξ_1, and ξ_0, and note that $\eta(\xi_0) = \eta(\xi_1)$. The proof is complete. □

4. Discussion

We have proved that for a flow with vorticity that is decreasing with depth all periodic steady waves are symmetric if their profile is a single-valued function which is monotonic between each crest and trough. In particular, this symmetry property holds for irrotational waves. Note that Garabedian (1965) proved the symmetry of waves with one local maximum and one local minimum per wavelength on every streamline except for the flat bottom. Toland (2000) gave a simplified argument for symmetry under the same restrictive hypothesis. In the irrotational case we only require that the streamline represented by the free surface has one local maximum and one local minimum per wavelength. We therefore recover in the case of irrotational flow the improved version of Garabedian's result presented in Okamoto & Shoji (2001). While our approach has similarities, in its use of maximum principles and in how it exploits reflection methods, to that of Okamoto & Shoji (2001), our theorem is a genuine improvement upon the irrotational case. Indeed, our theorem applies to a large class of flows with non-constant vorticity whereas the proof of Okamoto & Shoji (2001) can be adapted with no essential differences only to the case of constant vorticity.

Teles da Silva & Peregrine (1988) present numerical evidence of periodic overhanging wave profiles on flows of constant vorticity. It would be interesting to extend the present analysis by finding an approach that allows one to handle overhanging wave profiles. A further aspect of interest would be the case of arbitrary vorticity distributions, where recent investigations by Constantin & Strauss (2002, 2003) show the existence of symmetric steady periodic waves. The question of whether symmetry for such flows is generally guaranteed for gravity waves with monotonic profiles between crest and trough is only partially answered by our main result since a restriction on the maximal elevation of the water above the flat bottom is required for (1) to hold.

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REFERENCES

Amick, C. J. & Toland, J. F. 1981 On periodic water-waves and their convergence to solitary waves in the long-wave limit. *Phil. Trans. R. Soc. Lond.* A **303**, 633–669.

Baddour, R. E. & Song, S. W. 1998 The rotational flow of finite amplitude periodic water waves on shear currents. *Appl. Ocean Res.* **20**, 163–171.

Banner, M. L. & Peregrine, D. H. 1993 Wave breaking in deep water. *Annu. Rev. Fluid Mech.* **25**, 373–397.

Constantin, A. 2001 On the deep water wave motion. *J. Phys.* A **34**, 1405–1417.

Constantin, A. & Strauss, W. 2002 Exact periodic traveling water waves with vorticity. *C. R. Acad. Sci. Paris* **335**, 797–800.

Constantin, A. & Strauss, W. 2004 Exact steady periodic water waves with vorticity. *Commun. Pure Appl. Maths* (in press).

Crapper, G. 1984 *Introduction to Water Waves*. Ellis Horwood, Chichester.

Fraenkel, L. E. 2000 *An Introduction to Maximum Principles and Symmetry in Elliptic Problems*. Cambridge University Press.

Garabedian, P. 1965 Surface waves of finite depth. *J. d'Anal. Math.* **14**, 161–169.

Gerstner, F. 1809 Theorie der Wellen samt einer daraus abgeleiteten Theorie der Deichprofile. *Ann. Physik* **2**, 412–445.

Gidas, B., Ni, W. M. & Nirenberg, L. 1979 Symmetry and related properties via the maximum principle. *Commun. Math. Phys.* **68**, 209–243.

Johnson, R. S. 1997 *A Modern Introduction to the Mathematical Theory of Water Waves*. Cambridge University Press.

Jonsson, I. G. 1990 Wave-current interactions. In *The Sea* (ed. B. Le Mehaute & D. M. Hanes), Vol. 9, pp. 65–120. Wiley-Interscience.

Keady, G. & Norbury, J. 1978 On the existence theory for irrotational water waves. *Math. Proc. Camb. Phil. Soc.* **83**, 137–157.

Lighthill, J. 1978 *Waves in Fluids*. Cambridge University Press.

Longuet-Higgins, M. S. 1953 Mass transport in water waves. *Phil. Trans. R. Soc. Lond.* A **245**, 535–581.

Longuet-Higgins, M. S. 1960 Mass transport in the boundary layer at a free oscillating surface. *J. Fluid Mech.* **8**, 293–306.

Okamoto, H. & Shoji, M. 2001 *The Mathematical Theory of Permanent Progressive Water-Waves*. World Scientific.

Peregrine, D. H. 1976 Interaction of water waves and currents. *Adv. Appl. Mech.* **16**, 9–117.

Serrin, J. 1971 A symmetry property in potential theory. *Arch. Rat. Mech. Anal.* **43**, 304–318.

Smith, R. 1976 Giant waves. *J. Fluid Mech.* **77**, 417–431.

Swan, C., Cummins, I. P. & James, R. L. 2001 An experimental study of two-dimensional surface water waves propagating on depth-varying currents. *J. Fluid Mech.* **428**, 273–304.

Teles da Silva, A. F. & Peregrine, D. H. 1988 Steep, steady surface waves on water of finite depth with constant vorticity. *J. Fluid Mech.* **195**, 281–302.

Thomas, G. & Klopman, G. 1997 Gravity waves in water of finite depth. In *Wave-Current Interactions in the Nearshore Region*. Computational Mechanics Publications, Vol. 10, pp. 255–319, Southampton-Boston.

Toland, J. F. 2000 On the symmetry theory for Stokes waves of finite and infinite depth. In *Trends in Applications of Mathematics to Mechanics*. Monographs and Surveys in Applied Mathematics, Vol. 106, pp. 207–217. Chapman & Hall/CRC.